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An Inequality for Expected Values of Sample
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AN INEQUALITY FOR EXPECTED VALUES OF SAMPLE QUANTILES¹

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1. Introduction. Let F be a continuous distribution function on R^1 that is strictly increasing on the (finite or infinite) open interval I where $0 < F < 1$, and let G denote the inverse of F . For $n = 1, 2, \dots$ and $0 < \lambda < 1$, let

$$(1.1) \quad \gamma_n(\lambda) = [\Gamma(n+1)/\Gamma(\lambda(n+1))\Gamma((1-\lambda)(n+1))] \int_0^1 G(y) y^{\lambda(n+1)-1} \cdot (1-y)^{(1-\lambda)(n+1)-1} dy.$$

Obviously, if $X_{i:n}$ denotes the i th order statistic of a sample of size n from the parent distribution F , then

$$\gamma_n(i/(n+1)) = EX_{i:n}, \quad i = 1, 2, \dots, n.$$

We shall call $\gamma_n(\lambda)$ the expected value of the λ -quantile of a sample of size n from F , even though this interpretation is meaningless when $\lambda(n+1)$ is not an integer. We shall assume that for some λ the integral converges for sufficiently large n , which ensures that the same will hold for every $0 < \lambda < 1$. By making minor changes in W. Hoeffding's proof in [2], one shows that γ_n converges to G on $(0, 1)$ for $n \rightarrow \infty$.

Consider another continuous distribution function F^* that is strictly increasing on the interval I^* where $0 < F^* < 1$, and let G^* , γ_n^* and $X_{i:n}^*$ be defined for F^* analogous to G , γ_n and $X_{i:n}$ for F . Furthermore let

$$(1.2) \quad \phi(x) = G^*F(x), \quad x \in I.$$

In [5] the author studied the following order relations between F and F^* :

$$(1.3) \quad \phi \text{ is convex on } I;$$

$$(1.4) \quad F \text{ and } F^* \text{ represent symmetric distributions and } \phi \text{ is concave-convex on } I.$$

If x_0 denotes the median of F , relation (1.4) implies that ϕ is antisymmetric about x_0 (i.e. $\phi(x_0 + x) + \phi(x_0 - x) = 2\phi(x_0)$) and hence that ϕ is concave for $x < x_0$ and convex for $x > x_0$.

Let ϕ_n be the function that maps the expected value of the λ -quantiles of a sample of size n from F on the corresponding quantities for F^* :

$$(1.5) \quad \phi_n(x) = \gamma_n^* \gamma_n^{-1}(x).$$

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For $n \rightarrow \infty$, ϕ_n will converge to the function ϕ on I that maps the population quantiles of F on those of F^* . It is shown in this note that if relations (1.3) or (1.4) hold, ϕ_n shares the convexity or concave-convexity of ϕ , and the convergence of ϕ_n to ϕ is monotone. The convexity property yields a theorem on the behavior of the ratio of expected values of spacings of consecutive order statistics from F and F^* . Simple applications are given in Section 3.

2. The results.

THEOREM 2.1. *If condition (1.3) holds, $\phi_n(x)$ is convex in x for fixed n , and non-increasing in n for fixed x .*

PROOF. For each fixed n the densities

$$(2.1) \quad f_\lambda(y) = [\Gamma(n+1)/\Gamma(\lambda(n+1))\Gamma((1-\lambda)(n+1))] \cdot y^{\lambda(n+1)-1}(1-y)^{(1-\lambda)(n+1)-1}$$

constitute a one-parameter exponential family for $0 < \lambda, y < 1$, and consequently the family is strictly totally positive of order ∞ in λ and y (cf. [3]). According to a slight elaboration of a result due to S. Karlin that is given in [4], the convexity of ϕ_n follows from the definition of γ_n and γ_n^* , the total positivity of $f_\lambda(y)$, the monotonicity of F and the convexity of ϕ . Also

$$(2.2) \quad \gamma_n(\lambda) = \lambda\gamma_{n+1}(\lambda + (1-\lambda)/(n+2)) + (1-\lambda)\gamma_{n+1}(\lambda - \lambda/(n+2))$$

and the same holds for γ_n^* . This is easily verified by adding integrands in expression (1.1). Hence, because of the convexity of ϕ_{n+1} ,

$$\begin{aligned} & \phi_{n+1}\gamma_n(\lambda) \\ &= \phi_{n+1}(\lambda\gamma_{n+1}(\lambda + (1-\lambda)/(n+2)) + (1-\lambda)\gamma_{n+1}(\lambda - \lambda/(n+2))) \\ (2.3) \quad & \leq \lambda\phi_{n+1}\gamma_{n+1}(\lambda + (1-\lambda)/(n+2)) + (1-\lambda)\phi_{n+1}\gamma_{n+1}(\lambda - \lambda/(n+2)) \\ &= \lambda\gamma_{n+1}^*(\lambda + (1-\lambda)/(n+2)) + (1-\lambda)\gamma_{n+1}^*(\lambda - \lambda/(n+2)) \\ &= \gamma_n^*(\lambda), \end{aligned}$$

or, replacing $\gamma_n(\lambda)$ by x , $\phi_{n+1}(x) \leq \gamma_n^*\gamma_n^{-1}(x) = \phi_n(x)$.

In the same vein we have

THEOREM 2.2. *If condition (1.4) holds, $\phi_n(x)$ is antisymmetric concave-convex about x_0 for fixed n , and non-increasing in n for fixed $x > x_0$.*

PROOF. Obviously ϕ_n is antisymmetric about x_0 . Since ϕ is concave-convex, G^* is a concave-convex function of G and hence

$$h(y) = G^*(y) - a - bG(y)$$

can have at most three changes of sign on $(0, 1)$ for any a and b . If it does change sign three times, the signs occur in the order $(-, +, -, +)$ for increasing values of the argument. It follows from the variation diminishing property of totally positive kernels (cf. [3]) that

$$\gamma_n^*(\lambda) - a - b\gamma_n(\lambda) = \int_0^1 h(y)f_\lambda(y) dy$$

changes sign at most three times; if it does have three sign changes, the signs occur in the order $(-, +, -, +)$. Substituting $\gamma_n(\lambda) = x$ we find that $\phi_n(x) - a - bx$ possesses the same property for any a and b . A simple geometrical argument based on the antisymmetry of ϕ_n shows that this implies that ϕ_n is concave-convex about x_0 . Since for $\lambda > \frac{1}{2}$

$$(\lambda + (1 - \lambda)/(n + 2)) + (\lambda - \lambda/(n + 2)) > 1,$$

and hence by the antisymmetry of γ_{n+1}

$$\gamma_{n+1}(\lambda + (1 - \lambda)/(n + 2)) + \gamma_{n+1}(\lambda - \lambda/(n + 2)) > 2x_0$$

the inequality of (2.3) remains valid now that ϕ_n is antisymmetric and concave-convex instead of convex. This completes the proof.

We note that in the proofs of Theorems 2.1 and 2.2 we have only made use of the total positivity of $f_\lambda(y)$. Exploiting the fact that the total positivity is strict, one finds that the convexity (or concave-convexity) in x as well as the monotonicity in n of $\phi_n(x)$ are strict, unless ϕ is linear on I .

The quantities $\gamma_n(\lambda)$ for non-integer $\lambda(n + 1)$ were introduced to facilitate the discussion of λ -quantiles for fixed λ and varying n . However, in considering the convexity of ϕ_n for fixed n , we may as well restrict ourselves to the case where $i = \lambda(n + 1)$ is an integer. Theorem 2.1 then states that if condition (1.3) holds, i.e. if G^* is a convex function of G , then $EX_{i:n}^*$ is a convex function of $EX_{i:n}$ for varying i and fixed n , i.e.

$$(2.4) \quad (EX_{i+1:n}^* - EX_{i:n}^*)/(EX_{i+1:n} - EX_{i:n})$$

is non-decreasing in i for fixed n . We recall that the proof of this assertion rests solely on the fact that the family (2.1), which for $i = \lambda(n + 1)$ becomes

$$(2.5) \quad f_{i:n}(y) = [n!/(i - 1)!(n - i)!]y^{i-1}(1 - y)^{n-i},$$

is totally positive of order infinity in i and y for fixed n . However, the family (2.5) is also totally positive of order infinity in n and $(1 - y)$ for fixed i . One easily verifies that this implies that $EX_{i:n}^*$ is also a convex function of $EX_{i:n}$ for varying n and fixed i . Since $EX_{i:n}$ is decreasing in n for fixed i , it follows that

$$(EX_{i:n}^* - EX_{i:n+1}^*)/(EX_{i:n} - EX_{i:n+1})$$

is non-increasing in n . Using formula (2.2) for $\lambda(n + 1) = i$, i.e.

$$(2.6) \quad EX_{i:n} = [i/(n + 1)]EX_{i+1:n+1} + [(n + 1 - i)/(n + 1)]EX_{i:n+1},$$

and the corresponding expression for $EX_{i:n}^*$, we find

$$\begin{aligned} (EX_{i:n}^* - EX_{i:n+1}^*)/(EX_{i:n} - EX_{i:n+1}) \\ = (EX_{i+1:n+1}^* - EX_{i:n+1}^*)/(EX_{i+1:n+1} - EX_{i:n+1}), \end{aligned}$$

and hence (2.4) is non-increasing in n .

By considering the distribution functions $1 - F^*(-x)$ and $1 - F(-x)$ in-

stead of F and F^* one easily shows that

$$(2.7) \quad (EX_{n-i+1:n}^* - EX_{n-i:n}^*) / (EX_{n-i+1:n} - EX_{n-i:n})$$

is non-increasing in i and non-decreasing in n . The former conclusion is of course equivalent to the monotonicity in i of (2.4). We have proved

THEOREM 2.3. *If condition (1.3) holds, the quantities (2.4) are non-decreasing in i and non-increasing in n , whereas (2.7) is non-decreasing in n .*

We note that the last assertion of the theorem may also be proved directly by using the total positivity of (2.5) in i and y for fixed $(n - i)$ and applying (2.6).

It may be of interest to point out the similarity of Theorem 2.3 to inequalities that were recently obtained by R. E. Barlow and F. Proschan [1] for the case where $F(0) = F^*(0) = 0$ and ϕ is starshaped (i.e. $\phi(x)/x$ non-decreasing on I). By total positivity arguments similar to those given above they show that $EX_{i:n}^*/EX_{i:n}$ is non-decreasing in i and non-increasing in n , whereas $EX_{n-i:n}^*/EX_{n-i:n}$ is non-decreasing in n .

3. Applications. Let F be the uniform distribution function on $(0, 1)$. Then

$$\gamma_n(\lambda) = \lambda \quad \text{for } 0 < \lambda < 1,$$

$\phi = G^*$ and $\phi_n = \gamma_n^*$. If F^* is differentiable on I^* , it satisfies conditions (1.3) or (1.4) if its density $F^{*'} is non-increasing on I^* , or symmetric and unimodal respectively. Consequently we have:$

The expected value of the λ -quantile of a sample of size n from a distribution with non-increasing density is a non-increasing function of n ; if the density is symmetric and unimodal the conclusion remains valid for $\lambda > \frac{1}{2}$. Moreover, if $F^{*'}$ is non-increasing, $(n + 1)(EX_{i+1:n}^* - EX_{i:n}^*)$ is non-decreasing in i and non-increasing in n , whereas $(n + 1)(EX_{n-i+1:n}^* - EX_{n-i:n}^*)$ is non-decreasing in n .

As a second example consider the case where F^* denotes the exponential distribution function. Then condition (1.3) is satisfied if the distribution F has increasing failure rate

$$q(x) = F'(x)/(1 - F(x))$$

(cf. [1] or [5]). We have (cf. similar results in [1]): If F has increasing failure rate, then $(n - i)(EX_{i+1:n} - EX_{i:n})$ is non-increasing in i and non-decreasing in n , whereas $(EX_{n-i+1:n} - EX_{n-i:n})$ is non-increasing in n .

For other cases where relations (1.3) or (1.4) are satisfied and the results of this paper may be applied, the reader is referred to [5].

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